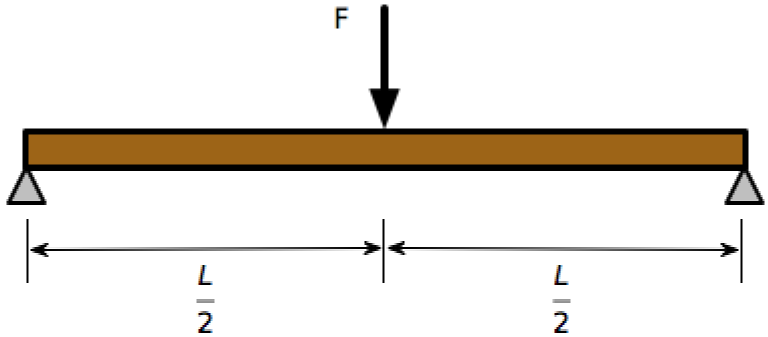
Solved Problems (Rayleigh Ritz method)

1.Rayleigh Ritz method applied to Beams

#1) Find the deflection at the centre using Rayleigh Ritz method for the simply supported beam subjected to Concentrated load at the centre



Solution:

Take y = $\sum_{n=1}^{k} a_n \sin\frac{n\pi x}{l}$.

According to Rayleigh Ritz method $\prod$ = U – W and $\frac{\partial \prod}{\partial C_n} = 0$ where π = Potential energy U = Strain energy and W is the external work done.

$\prod = \frac{1}{2}\int EI(\frac{d^2 y}{d^2 x})^2 dx - py_c$ where y_c is the deflection at the centre.

$$\frac{dy}{dx} = a_n \frac{n\pi}{l}\sum_{n=1}^{k} Cos\frac{n\pi x}{l}$$

$$\left(\frac{d^2 y}{dx^2}\right) = a_n \frac{n^2\pi^2}{l^2}\sum_{n=1}^{k} -\sin\frac{n\pi x}{l}$$

$$\left(\frac{d^2 y}{dx^2}\right)^2 = a^2{}_n \frac{n^4\pi^4}{l^4}\sum_{n=1}^{k} \sin^2\frac{n\pi x}{l}$$

$$\prod = \frac{1}{2}\int_0^l EIa^2{}_n \frac{n^4\pi^4}{l^4}\sum_{n=1}^{k}\sin^2\frac{n\pi x}{l}dx - P\sum_{n=0}^{K} a_n \sin\frac{n\pi}{2} \quad \text{--------(1.a)}$$

We Know that $\int_0^l \sin\frac{n\pi x}{l}\sin\frac{m\pi x}{l}dx = 0$ when ***m*** is not equal to ***n*** .

$\int_0^l \sin\frac{n\pi x}{l}\sin\frac{m\pi x}{l}dx = \frac{l}{2}$ when **m** = **n** Hence equation 1.a becomes,

$$\frac{\partial \prod}{\partial a_n} = \frac{1}{2} EI\pi^4 2a_n \frac{\pi^4}{l^4}\frac{l}{2} - PSin\frac{n\pi}{2} \text{ --------------(1b)}$$

From 1 b eqn, $a_n = \frac{2pl^3}{\pi^4 EI}\frac{1}{n^4}\sin\frac{n\pi}{2}$

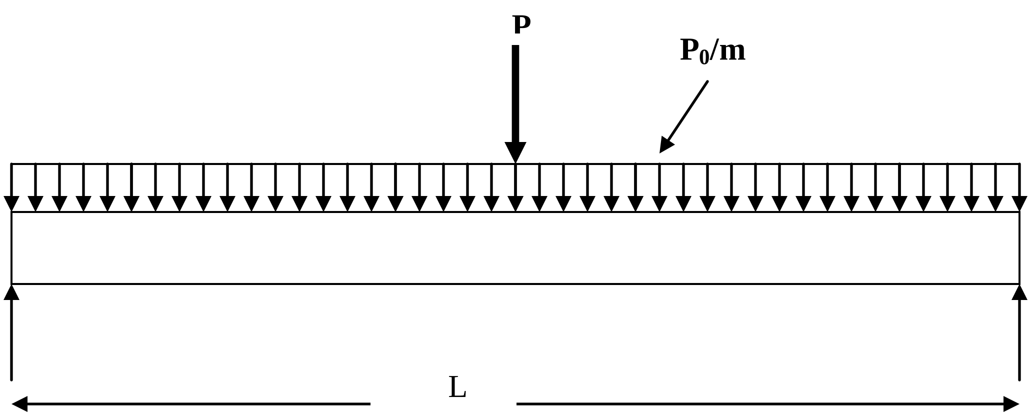
There fore $y = \frac{2pl^3}{\pi^4 EI}\sum_{n=1,3,5...}^{k}\frac{1}{n^4} Sin\frac{n\pi}{2} Sin\frac{n\pi x}{l}$ ------------(1c)

To find out y_c substitute x = l/2. in above equation (1.c)

$$y_c = \frac{2pl^3}{\pi^4 EI}\sum_{n=1,3,5...}^{k}\frac{1}{n^4}$$

Hence we have $y_c = \frac{2pl^3}{\pi^4 EI}\left(1+\frac{1}{3^4}+\frac{1}{5^4}\right)$ which is the deflection at the centre.

#2) Find the maximum deflection of a simply supported beam subjected to central concentrated load P at the centre and uniformly distributed load P_0/m through out the beam



Let $y = a_1 \sin\frac{\pi x}{l} + a_2 \sin\frac{3\pi x}{l}$

$$U = \frac{EI}{2}\int_0^L\left(\frac{d^2y}{dx^2}\right)^2 dx$$

$$\frac{dy}{dx} = \frac{\pi}{L} a_1 Cos\frac{\pi x}{L} + a_2\frac{3\pi}{L} Cos\frac{3\pi x}{L}$$
$$\frac{d^2y}{dx^2} = -\frac{\pi^2}{L^2} a_1 Sin\frac{\pi x}{L} - a_2\frac{9\pi^2}{L^2} Sin\frac{3\pi x}{L}$$

$U = \frac{EI}{2}\int_0^L \left(\frac{d^2y}{dx^2}\right)^2 dx = \frac{EI}{2}\int_0^L \left(\frac{-\pi^4 a_1}{L^2} Sin\frac{\pi x}{L} - \frac{a_2\pi^2 9}{L^2} Sin\frac{3\pi x}{L}\right)^2 dx$ which yields

U = $\frac{EI\pi^4}{L^4}(a_1{}^2 + 81a_2^2)$

Now W = $\int_0^L P_0 y dx + Py_{\max}$

$y_{max} = a_1 - a_2$

Hence W = $\left[\int_0^L P_0\left(a_1 Sin\frac{\pi x}{L} + a_2 Sin\frac{3\pi x}{L}\right)dx\right] + P(a_1 - a_2)$

W = $2P_0\frac{L}{\pi}\left[a_1 + \frac{a_2}{3}\right]$ + $P(a_1 - a_2)$

$\prod$ = U – W

$\prod = \frac{EI\pi^4}{4L^3}(a_1{}^2 + 81a_2{}^2) - \frac{2P_0L}{\pi}(a_1 + \frac{a_2}{3}) - P(a_1 - a_2)$

$\frac{\partial\prod}{\partial a_1} = 0$ and $\frac{\partial\prod}{\partial a_2} = 0$ which gives the following equations:

$$\frac{EI\pi^4}{2L^3}a_1 - \frac{2P_0L}{\pi} - P = 0$$
$$\frac{81EI\pi^4 a_2}{2L^3} - \frac{2P_0L}{3\pi} + P = 0$$

From the above equations we get $a_1 = \frac{2L^3}{EI\pi^4}(\frac{2P_0L}{3\pi} + P)$ and

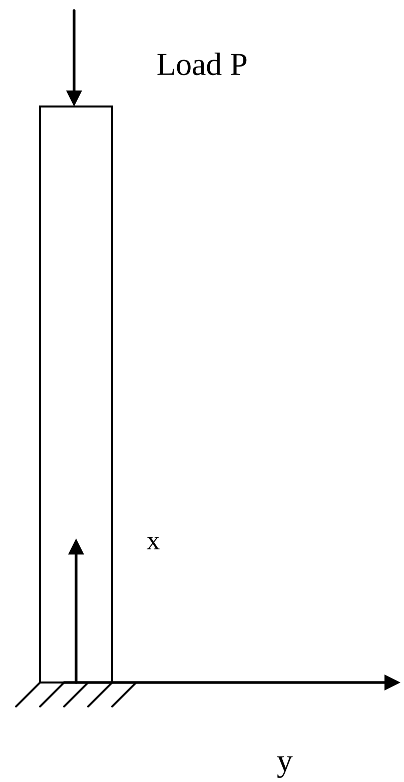
$a_2 = \frac{2L^3}{81EI\pi^4}(\frac{2P_0L}{3\pi} - P)$

$y_{max} = a_1 - a_2$

$y_{max} = \frac{PL^3}{48EI} + \frac{P_0L^4}{76.82EI}$

2.Rayleigh Ritz Method applied to Columns:

#3) A Uniform Column is fixed at the bottom and free at the top. It carries a compressive load at the free end. Investigate the critical load of the column by assuming a second degree polynomial as
$y = a_0 + a_1 x + a_2 x^2$.



Take the origin at the fixed end.
At x = 0, y = 0
At x = 0, dy/dx = 0

The above conditions give $a_0 = a_1 = 0$ and hence
$y = a_2 x^2$
When x = L, y = q
$q = a_2 L^2$ and so we have $a_2 = q/L^2$

Hence $y = qx^2/L^2$

$$\prod = \frac{1}{2}\int_0^L EI\left(\frac{d^2y}{d^2x}\right)^2 dx - \int_0^L \frac{P}{2}\left(\frac{dy}{dx}\right)^2 dx$$

$$\frac{dy}{dx} = \frac{2qx}{l^2} \text{ and } \left(\frac{d^2y}{dx^2}\right) = \frac{2q}{l^2}$$

$$\prod = \frac{1}{2}\int_0^L EI\left(\frac{2q}{L^2}\right)^2 dx - \int_0^L \frac{P}{2}\left(\frac{2qx}{L^2}\right)^2 dx \text{ which gives}$$

$$\Pi = \frac{2EIq^2}{L^3} - \frac{2pq^2}{3L}$$

$$\frac{\partial \prod}{\partial q} = 0 = \frac{4EIq}{L^3} - \frac{4pq}{3L} \text{ and this equation yields } P = \frac{3EI}{L^2}$$

Repeat the above problem by using $y = a_0 + a_1 x + a_2 x^2 + a_3 x^3$ and satisfying the bending moment condition at the top

When x = 0 , y = 0
Hence $a_0 = 0$.
When x = 0 dy/dx = 0 and we get $a_1 = 0$

When x = l, $\frac{d^2y}{dx^2} = 0$.

$$\frac{dy}{dx} = a_1 + 2a_2 x + 3a_3 x^2 \text{ and } \frac{d^2y}{dx^2} = 2a_2 + 6a_3 x$$

At x = l, $\frac{d^2y}{dx^2}=0$. Hence $\frac{d^2y}{dx^2}=2a_2+6a_3x$ =0 and a_2 = -3a_3l

So y = $-3a_3lx^2+a_3x^3=a_3(x^3-3lx^2)$

At x = l,y = q

There fore q= $a_3(l^3-3l^3)$ which gives $a_3=-\frac{q}{2l^3}$

Hence y = $-\frac{q}{2l^3}(x^3-3lx^2)$ and $\frac{dy}{dx}=\frac{q}{2l^3}(6lx-3x^2)$; $\frac{d^2y}{dx^2}=\frac{q}{2l^3}(6l-6x)$

$$\prod=\frac{1}{2}\int_0^l EI(\frac{d^2y}{d^2x})^2dx-\int_0^l\frac{P}{2}(\frac{dy}{dx})^2dx$$

$\Pi=\frac{EI}{2}\int_0^l\frac{q^2}{4l^6}(6l-6x)^2dx-\frac{P}{2}\int_0^l\frac{q^2}{4l^6}(6lx-3x^2)^2dx$ On simplifying

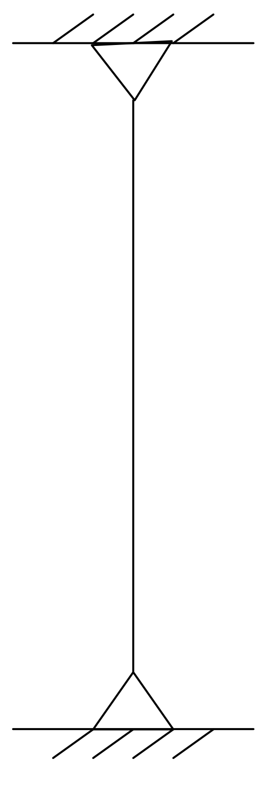
$\Pi=\frac{3EIq^2}{2l^3}-\frac{3pq^2}{5l}$.

$\frac{\partial\Pi}{\partial q}=\frac{3EIq}{l^3}-\frac{6pq}{5l}=0$ From this equation we get P=$\frac{15EI}{6l^2}=2.5\frac{EI}{l^2}$

#4) A uniform column hinged at both ends is subjected to a compressive load P at the two ends. Find the critical load using Rayleigh Ritz method if the trial function is

i) $y=\frac{4hx(l-x)}{l^2}$

ii) $y=aSin\frac{\pi x}{l}$



Solving (i)

$$\prod=\frac{1}{2}\int_0^L EI(\frac{d^2y}{d^2x})^2dx-\int_0^L\frac{P}{2}(\frac{dy}{dx})^2dx$$

$y=\frac{4hx(l-x)}{l^2}$ and so we have $\frac{dy}{dx}=\frac{4h}{l^2}(l-2x)$; $\frac{d^2y}{dx^2}=-\frac{8h}{l^2}$

$$\Pi=\frac{EI}{2}\int_0^l\frac{64h^2dx}{l^4}-\frac{P}{2}\int_0^l\frac{16h^2}{l^4}(l^2+4x^2-4lx)dx$$

$\frac{\partial\Pi}{\partial h}=\frac{64EIh}{l^3}-\frac{16ph}{3l}=0$. This equation gives the critical load.P_{cr}=$\frac{12EI}{l^2}$

Solving (ii)

$y = aSin\frac{\pi x}{l}$. This provides $\frac{dy}{dx} = \frac{a\pi}{l}Cos\frac{\pi x}{l}$ and $\frac{d^2y}{dx^2} = \frac{-a\pi^2}{l^2}Sin\frac{\pi x}{l}$

$$\prod = \frac{1}{2}\int_0^L EI(\frac{d^2y}{d^2x})^2 dx - \int_0^L \frac{P}{2}(\frac{dy}{dx})^2 dx$$

$$\Pi = \frac{EI}{2}\int_0^l \frac{a^2\pi^4}{l^4}Sin^2\frac{\pi x}{l}dx - \frac{P}{2}\int_0^l \frac{a^2\pi^2}{l^2}Cos^2\frac{\pi x}{l}dx$$ **---A**

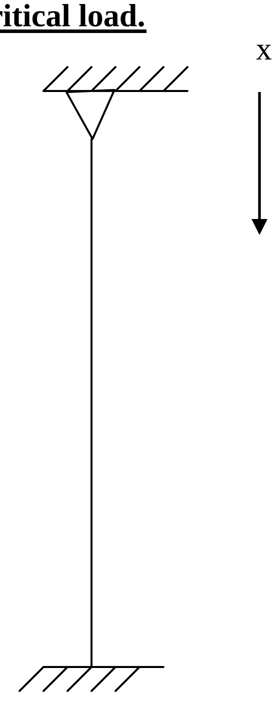
$$\int_0^l Sin^2\frac{\pi x}{l}dx = \int_0^l \frac{1 - Cos\frac{2\pi x}{l}}{2}dx = \frac{l}{2}$$

$$\int_o^l Cos^2\frac{\pi x}{l}dx = \int_0^l \frac{1 + Cos\frac{2\pi x}{l}}{2}dx = \frac{l}{2}$$

Substituting the above results in the equation A and finding $\frac{\partial\Pi}{\partial a} = \frac{2EI\pi^4 a}{2l^4}.\frac{l}{2} - \frac{P}{2}.\frac{2a\pi^2}{l^2}.\frac{l}{2} = 0$

The above expression yields critical Load Pcr= $\frac{\pi^2 EI}{l^2}$

#5) A uniform column is fixed at one end and is kept on rollers at the other end. In other words one end is fixed and other end is hinged. The length is 1m. Find the critical load.



At x = 0, y=0; --------(1)
At x = 1, y = 0 ; -----(2)
At x = 1, $\frac{dy}{dx} = 0$ -----(3)

Let y = $A_0 + A_1x + A_2x^2 + A_3x^3 + A_4x^4$. On applying (1), $A_0 = 0$

When x = 1, y = 0. This implies that $A_1+A_2+A_3+A_4 = 0$-----(4)

When x = 1, $\frac{dy}{dx} = 0$

Hence $\frac{dy}{dx} = A_1 + 2A_2 + 3A_3 + 4A_4 = 0$ ---------------------------(5)

(4) – (5) Implies that $-A_2 - 2A_3 - 3A_4 = 0$ And so $A_2 = -(2A_3 + 3A_4)$.------(6)
Substitute (6) in (4) so that we get $A_1 = A_3 + 2A_4$ ------------------------------(7)
Hence y = $(A_3 + 2A_4)x - (2A_3 + 3A_4)x^2 + A_3x^3 + A_4x^4$
y = $A_3(x - 2x^2 + x^3) + A_4(2x - 3x^2 + x^4)$ Put A_3 = a and A_4 = b
y= $a(x - 2x^2 + x^3) + b(2x - 3x^2 + x^4)$

$\frac{dy}{dx} = a(1-4x+3x^2)+(b(2-6x+4x^3)$ and $\frac{d^2y}{dx^2} = a(-4+6x\ )+b(-6+12x^2)$ -------(8)

$\prod = \frac{1}{2}\int_0^L EI(\frac{d^2y}{d^2x})^2 dx - \int_0^L \frac{P}{2}(\frac{dy}{dx})^2 dx$ ---(9)

Substituting (8) in (9)

$\Pi = \left[1680a^2 + 7056b^2 + 6720ab\right] - P\left[56a^2 + 288b^2 + 252ab\right]$ ----------------------------(10)

$\frac{\partial \Pi}{\partial a} = 0 and \frac{\partial \Pi}{\partial b} = 0$ ---(11)

Equation 11 gives us the following two results:

$a[3360-112P]+b[6720-252P]=0$ ---(12)

$a[6720-252P]+b[14112-576P]=0$ ---(13)

These are linear homogeneous equations. We know a and b cannot be equal to zero.
If they are zero they will not be buckled form. Therefore to get trivial solution the following must be true

$$\begin{vmatrix} (3360 - 112P) & (6720 - 252P) \\ (6720 - 252P) & (14112 - 576P) \end{vmatrix} = 0$$

Upon expanding and simplifying, we get

$1008\ P^2 - 129024\ P + 2257920 = 0$ which gives $P = (128 \pm 86.166)/2$

P= 20.92 is the lower value. During simplification EI was omitted. The same can be reintroduced.
The length 1m can also represented by “l”.

Hence $P = \frac{20.92EI}{l^2}$

References:

“Theory of Elasticity “ by Sadhu Singh

“Finite Element Primer” by V.K.Manickaselvam.